Classifying the complete arcs in small projective planes

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Overview

- Arcs in finite Desargeusian projective planes
- Canonical augmentation
- Large arcs only (unfinished business)

Arcs in finite Desarguesian projective planes

Definitions

A projective plane **P** consists of points and lines, such that

- Two points are joined by exactly 1 line
- Two lines intersect in exactly 1 point

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An arc $\mathcal{K}$ in **P** is a set of points such that

- No three points of $\mathcal{K}$ lie on the same line

Example: conic (ellipse, hyperbola, parabola)

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Goal: Find 'all' complete arcs in small projective planes.

Finite projective planes

If plane **P** is finite, then there are

- $q^2 + q + 1$ points and $q^2 + q + 1$ lines
- $q + 1$ points on each line, $q + 1$ lines through each point

q is the order of the projective plane.

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Maximum size of an arc $\mathcal{K}$

- $q + 2$ when q is *even*,
- $q + 1$ when q is *odd*.

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Desarguesian projective plane

Let **F** be a field.

- Points: $(x : y : z)$, $x, y, z \in \mathbf{F}$ (not all zero)
- Lines: $(A : B : C)^T$, $A, B, C \in \mathbf{F}$ (not all zero)

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- Conic: points satisfying $y^2 = xz$, i.e.,

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Today: field is finite with q elements
(= order of Desarguesian plane)

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When q is not prime: *algebraic extension* of prime field.

Example $q = 9 = 3^2$

- Elements are $0, \pm 1, \pm i, \pm 1 \pm i$
- Arithmetic modulo 3, $i^2 = -1$

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Symmetries

Let $M \in \mathbf{F}^{3\times 3}$, M nonsingular. The map

$$\begin{aligned} (x:y:z) &\mapsto (x:y:z)M, \\ (A:B:C)^T &\mapsto M^{-1}(A:B:C)^T \end{aligned}$$

is a projectivity. There are $\approx q^8$ projectivities.

(Examples: rotations, mirror symmetries, translations, ...)

A projectivity is an *automorphism* of **P** (incidence is preserved).

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Goal: Find all complete arcs in Desarguesian projective planes up to projective equivalence.

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Generation up to equivalence

Earlier results

- For $q \leq 9$. By hand (and mathematical theory).
- For $11 \leq q \leq 19$. By computer using *ad hoc* methods.

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What next:

- For $q > 29$ there are too many results.

  (For $q = 29$ there are $> 12 \cdot 10^9$ non-equivalent complete arcs. Computations took ≈ 5 years.)

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Canonical augmentation

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Depends on the selection of an appropriate choice function F on **P** which chooses a point in every non-empty set S:

$$F : S \mapsto F(S) \in S.$$

- What is the 'best' choice function depends on the problem
- **Important**: $F(S)$ should be easily computed for most S

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From F we derive a canonical dismantling:

$$S \to S' \to S'' \to \cdots \to \emptyset,$$

with $S' = S - \{F(S)\}, S'' = S' - \{F(S')\}, \ldots$

Canonical augmentation — simplified (cntd.)

Canonical augmentation reverses the dismantling process

$$\emptyset \to \cdots \to S'' \to S' \to S$$

- Start with the empty set,
- At each stage *augment* T to $T \cup \{t\}$ with $t \notin T$,
- Only do this when $T \cup \{t\}$ dismantles to T.

This guarantees that every set is generated only once !

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This is too simple...

Canonical augmentation — up to equivalence

We need all sets only up to equivalence.

F must be group invariant:

$$F(S^\phi) = F(S)^\phi, \quad \text{for all projectivities } \phi$$

- Generic example does no longer work
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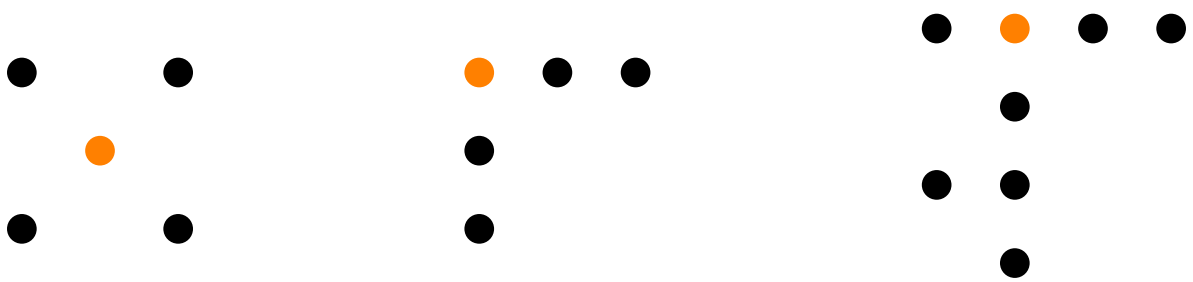
Typical example: F selects the point of largest 'degree'



Orbits — not points

A group invariant choice function does not always exist.

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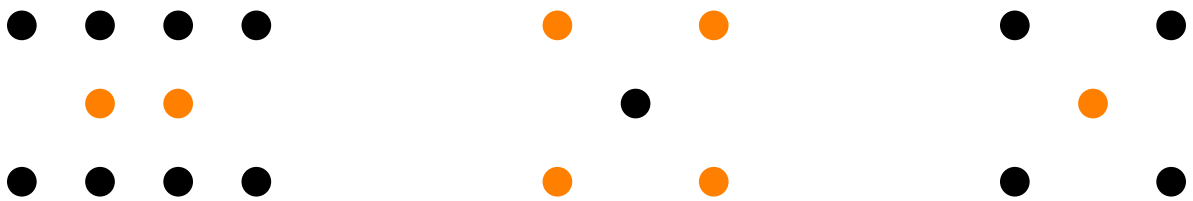
Solution: F chooses an orbit of the symmetry group of S

$$F : S \mapsto F(S) \in G_S \backslash\backslash S$$

$F(S)$ is a subset of S



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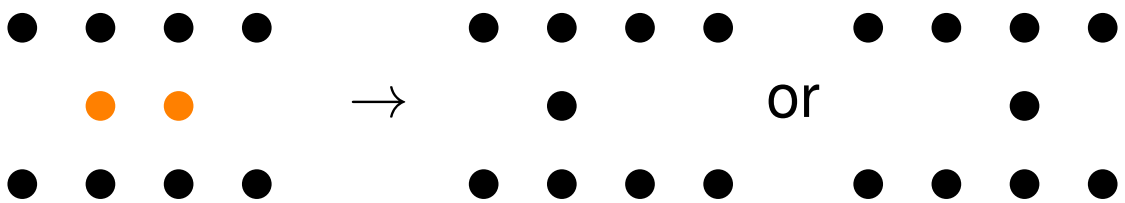
Canonical dismantling

The ‘parent’ is no longer unique

$$S' = S - \{s\}, \quad \text{with } s \in F(S).$$



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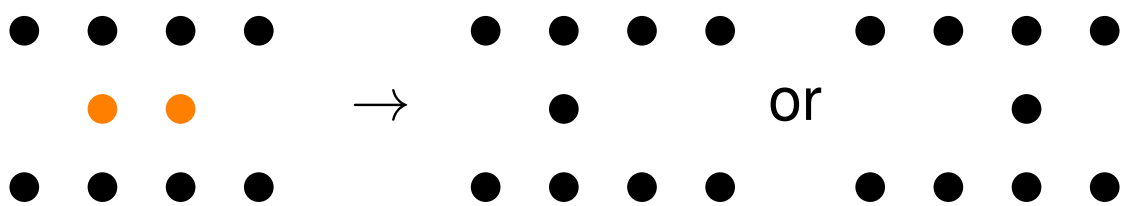

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- Parents of projective equivalent sets are projectively equivalent

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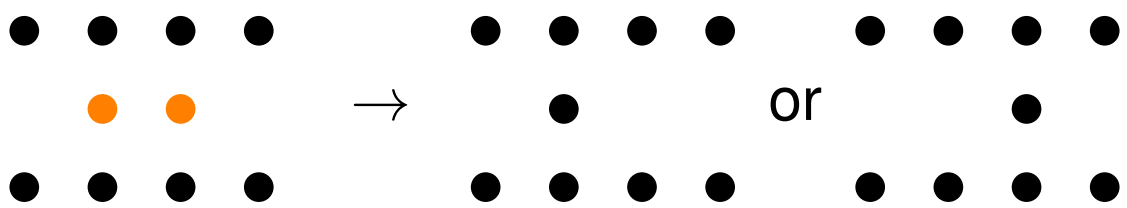

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Provides a canonical dismantling of *orbits* of sets.

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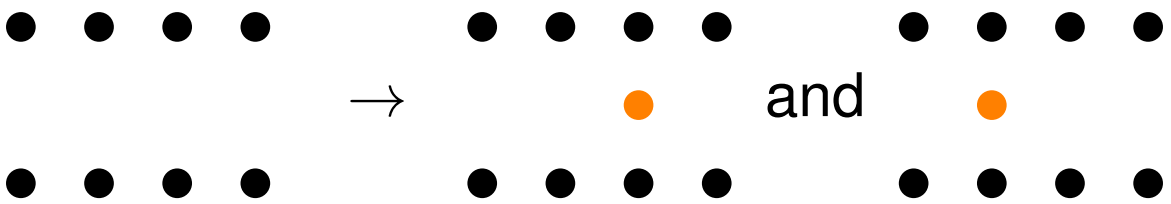

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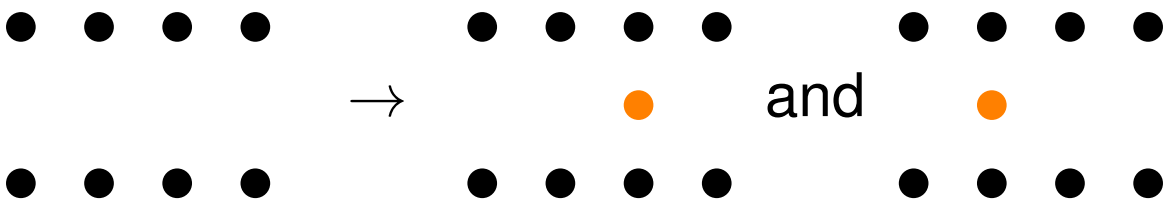

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- Add only one t for each orbit of $G_T \backslash\backslash T$
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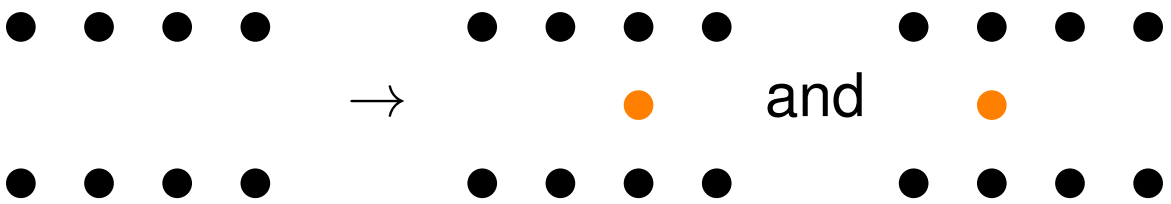

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Requires computing with groups!

How to avoid computing with groups

Every group invariant partitions the set S in unions of group orbits.

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Strategy:

- Make F prefer singletons
- If no singletons exist for a specific S, try alternative choice function(s) for that S
- If all else fails: use slow *fallback* choice function.

Choice function for arcs

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Lines intersect $\mathcal{K}$ in 0, 1 or 2 points (*external* line, *tangent* of *bisecant*).

Degree $d(p)$ of a point p: number of bisecants through p

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Line invariant and point invariant:

$$I(L) = \sum_{p \in L} d(p)^*, \quad I(p) = \sum_{L \ni p} I(L)^*,$$

where $\cdot^*$ is a simple *hash function*.

$I(p)$ **is a suitable invariant for defining** F

Final algorithm

Generate all arcs of size $k+1$ from all arcs of size k, *up to equivalence*:

- For each $\mathcal{K}'$ of size k,
- For all points $s \notin \mathcal{K}'$,
- Check that $\mathcal{K} = \mathcal{K}' \cup \{s\}$ is still an arc,
- Check that $\mathcal{K}$ dismantles to $\mathcal{K}'$,
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Do all of this depth first

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Results - Complete arcs of size n

	$q=5$	$q=7$	$q=8$	$q=9$	$q=11$	$q=13$	$q=16$	$q=17$
$n=6$	1	2	1	1				
$n=7$				1	1			
$n=8$		1		1	9	2		
$n=9$					3	29	2	
$n=10$			1	1	1	21	501	560
$n=11$							30	2644
$n=12$					1	1	9	553
$n=13$							1	8
$n=14$						1		1
$n=15$								
$n=16$								
$n=17$								
$n=18$							2	1

Results - Complete arcs of size n

	$q = 19$	$q = 23$	$q = 25$	$q = 27$	$q = 29$
$n = 10$	29	1			
$n = 11$	9541				
$n = 12$	30135	112449	606	7	
$n = 13$	2232	4341514	4072545	221429	708
$n = 14$	70	1828196	29151431	106320273	171139332
$n = 15$		58361	5709597	198631499	7402140892
$n = 16$		564	124577	20335114	4776509549
$n = 17$		5	434	276112	271929757
$n = 18$			41	950	2457679
$n = 19$				5	4190
$n = 20$	1				57
$n = 21$			1		2
$n = 22$				1	
$n = 23$					
$n = 24$		1			1
$n = 25$					
$n = 26$			1		
$n = 27$					
$n = 28$				1	
$n = 29$					
$n = 30$					1
CPU time	$\leq$ 1 hour	1 day	10 days	33 days	5 years

Large arcs only (unfinished business)

Large arcs

(Henceforth q is odd.)

Theorem [Segre 1954] All arcs of size $q + 1$ are conics

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What about size $q-1$?

- For $q = 7, 9, 11, 13$, examples exist that do not lie on a conic.
- **Theorems [1990s]** For $q > 83$ all lie on a conic
- Unknown cases:
  $q = 37, 41, 43, 47, \underline{49}, 53, 59, 61, 67, 71, 73, 79, \underline{81}, 83$

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Can we settle this by computer?

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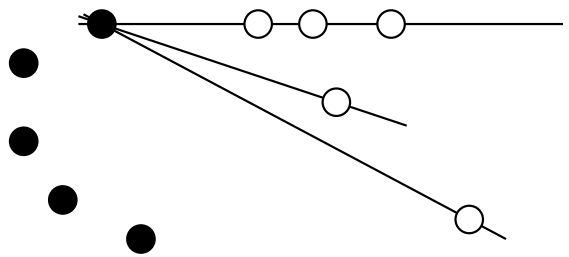
Adapting the earlier algorithm

When extending $\mathcal{K}'$ to $\mathcal{K}$, make sure that

- There are enough free points left ($d(p) = 0$)
- There are enough free tangents left



These adaptations make $q = 31$ just feasible, but not $q \geq 37$.

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The Lemma of Tangents

Let $\mathcal{K}$ denote an arc of size $q+2-t$. (Our case: $t=3$.)
Then every point of $\mathcal{K}$ lies on t tangents.

Theorem (Lemma of tangents) [Segre 1950s]
Assume $\mathcal{K}$ contains

$$e_1 : (1:0:0), \quad e_2 : (0:1:0), \quad e_3 : (0:0:1).$$

Assume the tangents through e_1, e_2, e_3 have equations

$$y = A_i z, \quad z = B_i x, \quad x = C_i y.$$

Then

$$\prod_{i=1}^{t} A_i B_i C_i = -1.$$

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Four triples in fixed quadruple of points

- Yield four equations
- However: not linearly independent
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Let $t = 3$, $|\mathcal{K}| = n$. Then there remain (at least)

$$\frac{1}{2}n(9 - n) - 1$$

degrees of freedom.

Generation strategy 1

For $n = 10$ we expect -6 degrees of freedom.

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- For every arc S of size 9 (up to equivalence)
- Run through all free points s
- Solve the equations for the 10 points of $S \cup \{s\}$
- Keep only those s for which there exists a solution
- Hopefully there are at least $q - 10$ of those.

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Not feasible: Even generating all S already takes too much time for $q = 83$.

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Generation strategy 2

For $n = 9$ we expect -1 degrees of freedom.

Algorithm 2:

- For every arc S of size 8 (up to equivalence)
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- Solve the equations for the 9 points of $S \cup \{s\}$
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Is this feasible for $q = 83$?

- Nr. of arcs of size 8: $\approx 8 \cdot 10^{10}$
- Nr. of free points per arc: $\approx q^2$
- Nr. of systems of equations to solve: $\approx 5 \cdot 10^{14}$

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Still not feasible: 250–500 years of computer time

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External points with 6 tangents

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Case 2: Arc contains at least one point with ≥ 6 tangents.

(Leads to next algorithm)

Generation strategy 3

For $n = 7$ we expect 6 degrees of freedom.

Algorithm 3

- For every arc S of size 6 (up to equivalence)
- Run through all free points p
- and through all (remaining) free points s
- Solve the equations for the 7 points of $S \cup \{s\}$
- Substitute the 6 known variables
- Partition the s according to the solutions

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Is this feasible for $q = 83$?

- Nr. of arcs of size 6: 56361

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Is this feasible for $q = 83$?

- Nr. of arcs of size 6: 56361
- **Feasible**: took 450 days of computer time
- **Results**: as expected (also for smaller q)

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Case 1: Unfinished business

"The resulting arc must have additional regularity properties which should make it more easy to generate such arcs."

Easier said than done.

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- Every arc of size $q-1$ has $\binom{q-1}{8}$ subsets S of size 8. ($\binom{82}{8} = 3.6 \cdot 10^{10}$).
- Every arc will be generated approximately that many times.

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Possible solution:

Select a smaller collection of starting arcs of size 8 that still guarantees that all arcs of size $q-1$ will be generated (up to equivalence).

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Veronesian arcs

Every 5 points of an arc lie on a unique conic.

Definition An arc $\mathcal{K}$ is called Veronesian if no 6 points of $\mathcal{K}$ lie on the same conic.

Conjecture If $\mathcal{K}$ is a Veronesian arc, then

$$|\mathcal{K}| \leq \frac{1}{3}(\sqrt{q}+1)^2$$

(For $q = 83$: $|\mathcal{K}| \leq 34$.)

- Mathematical proof? (With higher upper bound?)
- Provable by computer? (Even for $q = 83$?)

Generating non-Veronesian arcs

Corollary If $\mathcal{K}$ is large enough it must contain at least one subset of 6 points that lie on a conic.

Adapt Algorithm 2:

- Use only non-Veronesian arcs of size 8 as starting configurations

(Not yet clear what would be the gain in speed.)

Further thoughts:

- $(q-1)-6$ is still large enough to be non-Veronesian.
- $\mathcal{K}$ contains **two** conical subarcs of size 6.
- Is it feasible to generate all such arcs of 12 points ?

All suggestions are welcome!