Enumeration of Strongly Regular Designs

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Outline

Outline

Coherent Configurations
General picture

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- Coherent configurations (CCs) are a central notion in Algebraic Graph Theory.
- Many combinatorial objects can be defined in terms of CCs:
  - Strongly regular graphs
  - Distance regular graphs
  - Steiner systems
  - Partial geometries
  - Maximal arcs in projective planes

Coherent Configurations

- A CC is finite set Ω together with a set of binary relations R_i on Ω satisfying the following:
- $\Omega^2 = \bigcup_i R_i$
- Id_Ω is the union of some relations
- $R_i^T = R_{i'}$
- For $(x, y) \in R_k$ there are p_{ij}^k points z with $(x, z) \in R_i$ and $(z, y) \in R_j$.

Coherent Configurations

Construction from Permutation Groups

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- Coherent configurations can be constructed from permutation groups as follows:
- Let (G, Ω) be a finite permutation group.
- Let R_i, $i = 1, 2, \ldots$ be the orbits of G acting on Ω^2
- Then $(\Omega, \{R_i\})$ form a coherent configuration
- We call these configurations *Schurian*, their existence is “natural”
- Non-Schurian CC's are more surprising/interesting

Coherent Configurations
Fibers

- The reflexive relations of a CC define a partition of the point set Ω
- $i \in \Omega_k$ if $(i, i) \in R_k$
- We call the parts of this partition *fibers*
- In the Schurian case these are just the orbits of the underlying group

Coherent Configurations
Type

- Each (basic) relation in a CC connects points from a fiber F_i to points from a fiber F_j (possibly the same)
- If we know the f fibers we can count how many relations there are between any two fibers
- This gives us an $f \times f$ matrix which we call the *type* of the CC
- Since this matrix is symmetric we only write the upper diagonal part

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Coherent Configuration

Type Example

- Let P, B be a Steiner triple system
- Each block is a triple of points
- Each pair of points appears in exactly one block
- If points and blocks are the elements, which natural binary relations can we define?

Coherent Configuration

Type Example

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- A point and a block are incident or non-incident - we get two relations
- Two blocks can be identical, disjoint, or intersecting in one point - three relations
- Two points are equal or not equal - two relations
- In fact, we get a CC of type

$$\begin{bmatrix} 2 & 2 \\ & 3 \end{bmatrix}$$

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Coherent Configurations

Association Schemes

- CC's with a single fiber are *association schemes*
- They have a well-developed theory (Bannai-Ito, Zieschang)
- Practical work has been done by Miyamoto-Hanaki: Classification for orders 1-30, 32-34, 38

Coherent Configurations
Classification

- Long term goal: Classification of coherent configurations with few vertices
- A naïve approach works up to 13 vertices:

1	1
2	2
3	4
4	10
5	15
6	38
7	57

8	143
9	230
10	497
11	777
12	1923
13	2908

- Similar work has been done by Shigezumi et al., however the results differ slightly

Coherent Configurations

Classification Strategy

We propose the following long-term strategy for the classification:

- Start with the catalogue of association schemes
- Build two-fiber configurations
- Build three-fiber configurations
- From here the construction of *n*-fiber configurations is purely mechanical

Outline

Strongly Regular Designs

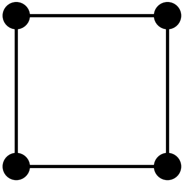
- One particular case of CC's with two fibers are those of type $\begin{pmatrix} 3 & 2 \\ & 3 \end{pmatrix}$
- They are equivalent to block designs with certain regularity properties
- They were called *strongly regular designs* by D. Higman
- Their axioms are self-dual

A strongly regular design with parameters n_i, S_i, a_i, b_i, N_i, P_i $(i = 1, 2)$ is a structure of points and blocks satisfying the following axioms and their duals:

- There are n_1 points
- Each block contains S_1 points
- There are exactly two block intersection sizes $a_1 > b_1$; we call two blocks adjacent if their intersection is of size a_1
- Given a point P and a block B, the number of points adjacent to P and incident to B is N_1 if P is incident to B, and P_1 otherwise

(The dual statements are formed by interchanging the roles of points and blocks, and by switching the indices 1 and 2)

- Consider the square



- Points are vertices; blocks are sides
- We get the following parameters:

#	n	S	a	b	N	P
1	4	2	1	0	1	1
	4	2	1	0	1	1

Enumerating Strongly Regular Designs

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- Now we concentrate on the constructive classification of SRD's

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Feasible Parameter Sets

- The intersection numbers of a SRD satisfy a lot of arithmetic conditions
- We enumerated feasible sets sorted by the number of blocks
- 22 sets with up to 20 blocks
- 107 set with up to 50 blocks
- 318 sets with up to 100 blocks

Feasible Parameter Sets

Small Examples

#	*n*	*S*	*a*	*b*	*N*	*P*
1	4	2	1	0	1	1
	4	2	1	0	1	1
2	8	4	2	0	3	3
	8	4	2	0	3	3
3	6	2	1	0	1	1
	9	3	1	0	2	1
4	9	3	1	0	2	2
	9	3	1	0	2	2
5	10	4	2	0	3	2
	10	4	2	0	3	2
6	8	4	2	0	2	2
	12	6	3	2	5	5

#	*n*	*S*	*a*	*b*	*N*	*P*
7	12	6	3	0	5	5
	12	6	3	0	5	5
8	12	6	3	2	4	4
	12	6	3	2	4	4
9	15	3	1	0	2	1
	15	3	1	0	2	1
10	8	2	1	0	1	1
	16	4	1	0	3	1
11	10	5	3	1	4	4
	16	8	4	0	6	4
12	12	3	1	0	2	2
	16	4	1	0	3	2

Outline

Construction of Designs

- Fix a set of parameters
- Start with a suitable point graph
- Find all potential blocks: induced subgraphs of order S_1 and valency N_1
- Find all sets of n_2 blocks with pairwise intersections a_1 and b_1

Construction of Designs
Implementation Details

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- The point graph is unique in the imprimitive case
- In the primitive case we use E. Spence's database of strongly regular graphs
- For the enumeration of subgraphs we use techniques for the orderly generation of sets (Read, Faradžev, Pech-R)
- Throughout we use GAP including Soicher's GRAPE and Design packages

Outline

Results

- We were able to complete the search for the first 33 parameter sets, with the exception of sets 26 and 32

#	n	S	a	b	N	P
26	24	12	6	4	10	10
	24	12	6	4	10	10
32	27	9	3	0	8	8
	27	9	3	0	8	8

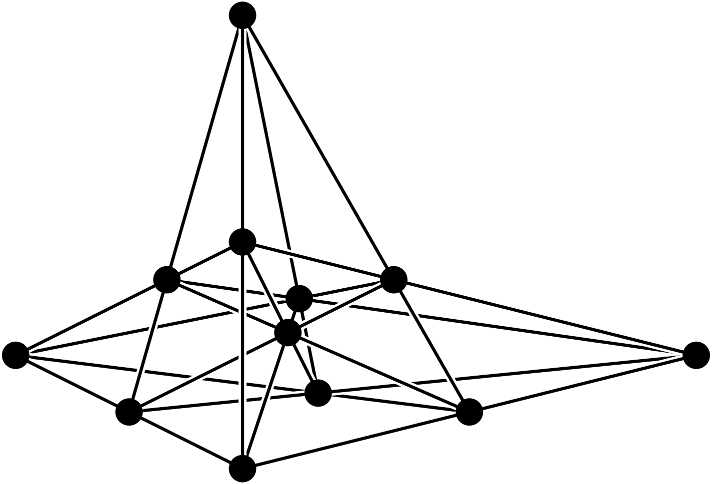
- For seven sets we did not find any solutions
- For eight sets we found a number of non-isomorphic solutions
- For all other parameter sets there are unique solutions

Outline

Reye's Configuration (Incidence Structure)

- Originally described by Th. Reye around 1880
- Configuration of points and lines
- Given in terms of a cube in projective 3-space
- Lines are edges and diagonals of the cube
- Points are intersections of lines
- Can be extended to planes
- 12 points, 16 lines, 12 planes

Reye's Configuration



Reye's Configuration

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- If we consider points and planes we get a a SRD with parameter set 8:

#	n	S	a	b	N	P
8	12	6	3	2	4	4
	12	6	3	2	4	4

- The geometrical picture shows it is invariant under Aut(cube) of order 48
- The full automorphism group is a lot bigger
- We will give a combinatorial description exposing the full automorphism group

- Start with the complete bipartite graph $\Gamma = K_{4,4}$
- Γ has 16 edges and $4! = 24$ 1-factors
- Its automorphism group is $G = S_2 \wr S_4$ of order $2 \cdot 24^2$
- Take the subgroup G^+ of even permutations
- Under G^+ the 1-factors split into two orbits of length 12

Reye's Configuration
A Combinatorial Construction

- Points: One orbit of 1-factors
- Lines: Edges of Γ
- Planes: The other orbit of 1-factors
- Incidence: "natural"
- The structure we obtain is isomorphic to Reye's configuration
- G^+ of order 576 is its full automorphism group.

Reye's Configuration
Non-schurian mates

- There are two more SRD's with parameter set #8
- These two are not schurian
- They have automorphism groups of size 24 and 32, respectively
- A friendly description of these objects is being elaborated

Outline

A Non-Schurian Strongly Regular Design

- Here we describe the unique design with parameter set #6:

#	n	S	a	b	N	P
6	8	4	2	0	2	2
	12	6	3	2	5	5

- It is the smallest non-schurian SRD; to our knowledge this structure hasn't been considered before.

A Non-Schurian Strongly Regular Design
Construction

- Consider the Cayley table of the group E_4 as a Latin square
- Consider Latin subsquares

a	*b*	*c*	*d*
b	*a*	*d*	*c*
c	*d*	*a*	*b*
d	*c*	*b*	*a*

- Count these subsquares!

A Non-Schurian Strongly Regular Design
Construction

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- Blocks are the 12 Latin subsquares
- Points are rows and columns of the original table
- Incidence is natural
- We get a strongly regular design
- Its full automorphism group has rank 11 (via theory of amorphic association schemes), hence it is non-schurian

Summary

- Strongly regular designs as a partial goal in the enumeration of coherent configurations
- All strongly regular designs with fewer than 24 blocks have been constructed

- Outlook
  - Improve the search algorithm by using the structure of the block graph
  - Canonical augmentation !?
  - Find additional numerical conditions for the existence of SRD's