A few families of non-Schurian association schemes [1]

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[1] Joint work with M. Klin

Preliminaries

Color graph

Under a color graph Γ we will mean an ordered pair $(V, \mathcal{R})$, where V is a set of vertices and $\mathcal{R}$ a partition of $V \times V$ into binary relations. The elements of $\mathcal{R}$ will be called as colors, and the number of colors is the rank of Γ.

Preliminaries

Coherent configuration

A coherent configuration is a color graph $\mathcal{N} = (\Omega, \mathcal{R})$, $\mathcal{R} = \{R_i \mid i \in I\}$, such that the following axioms are satisfied:

(i) The diagonal relation $\Delta_\Omega = \{(x,x) \mid x \in \Omega\}$ is a union of relations $\cup_{i \in I'} R_i$, for a suitable subset $I' \subseteq I$.

(ii) For each $i \in I$ there exists $i' \in I$ such that $R_i^T = R_{i'}$, where $R_i^T = \{(y,x) \mid (x,y) \in R_i\}$ is the relation transposed to R_i.

(iii) For any $i,j,k \in I$, the number $c_{i,j}^k$ of elements $z \in \Omega$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant depending only on i,j,k, and independent on the choice of $(x,y) \in R_k$.

Preliminaries

The numbers $c_{i,j}^k$ are called intersection numbers, or sometimes structure constants of $\mathcal{N}$.

An association scheme $\mathcal{N} = (\Omega, \mathcal{R})$ is a homogeneous coherent configuration, i.e. where the diagonal relation Δ_Ω does belong to $\mathcal{R}$.

A coherent configuration $\mathcal{N}$ is called commutative, if for all $i, j, k \in I$ we have $c_{ij}^k = c_{ji}^k$; and it is called symmetric if $R_i = R_i^T$ for all $i \in I$.

Preliminaries

The orbits of a group G on the set $\Omega \times \Omega$ are called 2-orbits, or orbitals.

If $2-\mathrm{Orb}(G, \Omega)$ is the set of 2-orbits of a permutation group (G, Ω), then $(\Omega, 2-\mathrm{Orb}(\Omega))$ is a coherent configuration. Those coherent configurations which can be obtained in this manner are called Schurian, otherwise non-Schurian. Thus, Schurian association schemes are coming from transitive permutation groups.

Preliminaries

The (combinatorial) group of automorphisms $\mathrm{Aut}(\mathcal{N})$ consists of permutations $\phi : \Omega \to \Omega$ which preserve the relations, i.e. $R_i^\phi = R_i$ for all $R_i \in \mathcal{R}$.

The color automorphisms preserve relations setwise, i.e. for $\phi : \Omega \to \Omega$ we have $\phi \in \mathrm{CAut}(\mathcal{N})$ if and only if for all $i \in I$ there exists $j \in I$ such that $R_i^\phi = R_j$.

An algebraic automorphism is a bijection $\phi : \mathcal{R} \to \mathcal{R}$ which satisfies $c_{ij}^k = c_{i^\phi j^\phi}^{k^\phi}$.

Preliminaries

Let G be a subgroup of the group of algebraic automorphisms of a coherent configuration. Let $\mathcal{R}/G$ denote the set of orbits of G on $\mathcal{R}$. For each $O \in \mathcal{R}/G$ define O^+ to be the union of all relations from O.

Then the set of relations $\{O^+ \mid O \in \mathcal{R}/G\}$ forms a coherent configuration on Ω. We will call it as algebraic merging of $\mathcal{R}$ with respect to G.

Why to study association schemes?

Applications

- codes
- designs
- statistical questions.

Algebra

- It is a nice table algebra.
- Transitive group actions on finite sets, distance-regular graphs, finite buildings can be viewed as association schemes.

Computer facilities

The main results are obtained as theoretical generalizations of observations, which were earned with the aid of a computer.

We used the computer algebra system GAP, in conjunction with GRAPE and nauty, packages COCO (Faradžev-Klin, 1991), COCO II (Reichard) and a package of elementary functions for association schemes on GAP (Hanaki, Miyamoto).

Computer facilities

COCO

was the first computer package for computing with coherent configurations developped in 1991 in Moscow by Faradžev's team.

- induced action of a permutation group on a combinatorial structure;
- the centralizer algebra of a permutation group;
- the intersection numbers;
- to find fusions;
- to calculate the (combinatorial) automorphism group.

Computer facilities

COCO II (S. Reichard)

- Works under GAP.
- New functions (color automorphisms, algebraic automorphisms, . . .).
- Still under construction.

Computer facilities

Webpage of Hanaki and Miyamoto

- `http://kissme.shinshu-u.ac.jp`
- Classification of association schemes with small number of vertices (< 39, but not $31, 35, 36, 37$).
- Elementary functions for association schemes on GAP.

Non-Schurian association schemes on 18 points

A careful analysis of known association schemes on 18 points, which are available from the homepage of Hanaki and Miyamoto, was our starting point.

The main interest was to understand two non-Schurian association schemes.

Non-Schurian association schemes on 18 points

$$\begin{pmatrix}
0&1&2&3&3&3&4&4&4&5&5&5&6&6&6&7&7&7\\
2&0&1&4&4&4&5&5&5&3&3&3&6&6&6&7&7&7\\
1&2&0&5&5&5&3&3&3&4&4&4&6&6&6&7&7&7\\
3&5&4&0&6&7&2&6&7&1&6&7&3&4&5&3&4&5\\
3&5&4&7&0&6&7&2&6&7&1&6&4&5&3&5&3&4\\
3&5&4&6&7&0&6&7&2&6&7&1&5&3&4&4&5&3\\
5&4&3&1&6&7&0&6&7&2&6&7&5&3&4&5&3&4\\
5&4&3&7&1&6&7&0&6&7&2&6&3&4&5&4&5&3\\
5&4&3&6&7&1&6&7&0&6&7&2&4&5&3&3&4&5\\
4&3&5&2&6&7&1&6&7&0&6&7&4&5&3&4&5&3\\
4&3&5&7&2&6&7&1&6&7&0&6&5&3&4&3&4&5\\
4&3&5&6&7&2&6&7&1&6&7&0&3&4&5&5&3&4\\
7&7&7&3&5&4&4&3&5&5&4&3&0&2&1&6&6&6\\
7&7&7&5&4&3&3&5&4&4&3&5&1&0&2&6&6&6\\
7&7&7&4&3&5&5&4&3&3&5&4&2&1&0&6&6&6\\
6&6&6&3&4&5&4&5&3&5&3&4&7&7&7&0&2&1\\
6&6&6&5&3&4&3&4&5&4&5&3&7&7&7&1&0&2\\
6&6&6&4&5&3&5&3&4&3&4&5&7&7&7&2&1&0
\end{pmatrix}$$

The color matrix of the non-Schurian association scheme on 18 points of rank 8 (nr. 62 in the catalogue).

Non-Schurian association schemes on 18 points

$$\begin{pmatrix}
0&1&1&2&2&2&3&3&3&4&4&4&5&5&5&5&5&5\\
1&0&1&5&5&5&3&3&3&4&4&4&2&2&2&5&5&5\\
1&1&0&5&5&5&3&3&3&4&4&4&5&5&5&2&2&2\\
2&5&5&0&3&4&2&5&5&2&5&5&1&3&4&1&3&4\\
2&5&5&4&0&3&5&2&5&5&2&5&4&1&3&4&1&3\\
2&5&5&3&4&0&5&5&2&5&5&2&3&4&1&3&4&1\\
4&4&4&2&5&5&0&1&1&3&3&3&5&2&5&5&5&2\\
4&4&4&5&2&5&1&0&1&3&3&3&5&5&2&2&5&5\\
4&4&4&5&5&2&1&1&0&3&3&3&2&5&5&5&2&5\\
3&3&3&2&5&5&4&4&4&0&1&1&5&5&2&5&2&5\\
3&3&3&5&2&5&4&4&4&1&0&1&2&5&5&5&5&2\\
3&3&3&5&5&2&4&4&4&1&1&0&5&2&5&2&5&5\\
5&2&5&1&3&4&5&5&2&5&2&5&0&3&4&1&3&4\\
5&2&5&4&1&3&2&5&5&5&5&2&4&0&3&4&1&3\\
5&2&5&3&4&1&5&2&5&2&5&5&3&4&0&3&4&1\\
5&5&2&1&3&4&5&2&5&5&5&2&1&3&4&0&3&4\\
5&5&2&4&1&3&5&5&2&2&5&5&4&1&3&4&0&3\\
5&5&2&3&4&1&2&5&5&5&2&5&3&4&1&3&4&0
\end{pmatrix}$$

The color matrix of the non-Schurian association scheme on 18 points of rank 6 (nr. 41 in the catalogue).

What was done?

Finally, we realized that, for each prime p, we can:

- work with an intransitive permutation group G of order p^3, acting on two orbits of length p^2,
- construct a corresponding coherent configuration $\mathcal{M}$ of rank $6p-2$ with two fibers,
- detect in $\mathcal{M}$ four association schemes.

Biaffine planes

- The biaffine plane $\mathcal{B}_p$ consists of two copies of $\mathbb{Z}_p \times \mathbb{Z}_p$: points $\mathcal{P}$ and "non-vertical" lines $\mathcal{L}$.
- Points: $P = [x, y]$.
- Lines: $\ell = (k, q)$, $y = k \cdot x + q$.
- Incidence: $P = [x, y]$ is incident to $\ell = (k, q)$ if and only if $y = k \cdot x + q$.

Biaffine planes

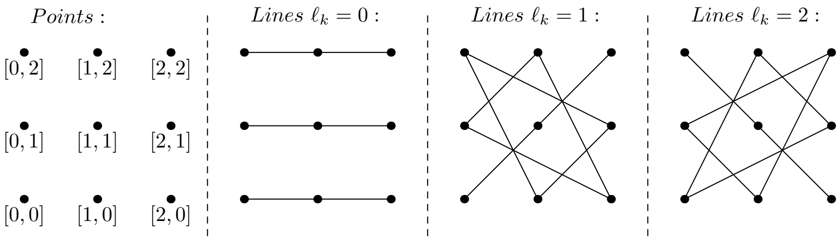


Figure: The objects of the biaffine plane $\mathcal{B}_3$.

The biaffine coherent configuration

Take an action of the permutation group $G = (\mathbb{Z}_p)^2 \rtimes \mathbb{Z}_p$ on the set $\Omega = \mathcal{P} \cup \mathcal{L}$.

At this stage it appears as *deus ex machina*.

The biaffine coherent configuration

$G \cong \langle t_{1,0}, t_{0,1}, \phi \rangle$, where

$$t_{a,b} : [x, y] \mapsto [x + a, y + b], \qquad (k, q) \mapsto (k, b + q - ak),$$

$$\phi : [x, y] \mapsto [x, y - x], \qquad (k, q) \mapsto (k - 1, q).$$

The biaffine coherent configuration

The group G has $6p-2$ orbits on $\Omega \times \Omega$:

- $(P_1, P_2) \in A_i \iff x_1 = x_2$ and $y_2 - y_1 = i$, where $i \in \mathbb{Z}_p$,
- $(P_1, P_2) \in B_i \iff x_2 - x_1 = i \neq 0$, where $i \in \mathbb{Z}_p \setminus \{0\}$,
- $(\ell_1, \ell_2) \in C_i \iff k_1 = k_2$ and $q_2 - q_1 = i$, where $i \in \mathbb{Z}_p$,
- $(\ell_1, \ell_2) \in D_i \iff k_2 - k_1 = i \neq 0$, where $i \in \mathbb{Z}_p \setminus \{0\}$,
- $(P_1, \ell_1) \in E_i \iff k_1 \cdot x_1 + q_1 - y_1 = i$, where $i \in \mathbb{Z}_p$,
- $(\ell_1, P_1) \in F_i \iff y_1 - k_1 x_1 - q_1 = i$, where $i \in \mathbb{Z}_p$.

The biaffine coherent configuration

Definition

The structure $\mathcal{M} = (\Omega, 2 - \mathrm{Orb}(G))$ is called as a biaffine coherent configuration.

Four color graphs

- $R_0 = A_0 \cup C_0$,
- $S_i = A_i \cup C_i$, where $i = 1, 2, \ldots, p-1$,
- $T_i = B_i \cup D_i$, where $i = 1, 2, \ldots, p-1$,
- $U_i = E_i \cup F_i$, where $i = 0, 1, 2, \ldots, p-1$.

- $S_i^* = S_i \cup S_{p-i}$,
- $T_i^* = T_i \cup T_{p-i}$,
- $U_i^* = U_i \cup U_{p-i}$.

- $S = S_1 \cup S_2 \cup \ldots \cup S_{p-1}$,
- $U = U_1 \cup U_2 \cup \ldots \cup U_{p-1}$.

Four color graphs

Color graph $\mathcal{M}_1$

colors: $R_0,\ S_1, \ldots, S_{p-1},\ T_1, \ldots, T_{p-1},\ U_0,\ U_1, \ldots, U_{p-1}$

Color graph $\mathcal{M}_2$

colors: $R_0,\ S_1^*,\ S_2^*, \ldots, S_{(p-1)/2}^*,\ T_1,\ T_2, \ldots, T_{p-1},\ U_0,\ U_1^*,\ U_2^*, \ldots, U_{(p-1)/2}^*$

Color graph $\mathcal{M}_3$

colors: $R_0,\ S,\ T_1,\ T_2, \ldots, T_{p-1},\ U_0,\ U$

Color graph $\mathcal{M}_4$:

colors: $R_0,\ S,\ T_1^*,\ T_2^*, \ldots, T_{(p-1)/2}^*,\ U_0,\ U.$

Theorem 1

Theorem 1

The following holds:

(a) $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ are association schemes.

(b) Combinatorial groups of automorphisms of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ contain a subgroup isomorphic to $G = \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$.

Groups of combinatorial automorphisms

Theorem 2

Let $\mathrm{Aut}(\mathcal{M}_1), \mathrm{Aut}(\mathcal{M}_2), \mathrm{Aut}(\mathcal{M}_3)$ and $\mathrm{Aut}(\mathcal{M}_4)$ are the combinatorial groups of automorphisms of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and $\mathcal{M}_4$, respectively. Then the followings hold:

(a) $\mathrm{Aut}(\mathcal{M}_1) \leq \mathrm{Aut}(\mathcal{M}_2) = \mathrm{Aut}(\mathcal{M}_3) \leq \mathrm{Aut}(\mathcal{M}_4)$,

(b) $|\mathrm{Aut}(\mathcal{M}_1)| = p^3$,

(c) $|\mathrm{Aut}(\mathcal{M}_2)| = 2p^3$,

(d) $|\mathrm{Aut}(\mathcal{M}_3)| = 2p^3$,

(e) $|\mathrm{Aut}(\mathcal{M}_4)| = 8p^3$.

Corollary

Corollary 3

For each $p > 3$ there exist at least four non-Schurian association schemes $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$, and $\mathcal{M}_4$ with ranks $3p-1$, $2p$, $p+3$, and $(p+7)/2$, respectively.

The second model of $\mathcal{M}$

Let

$$V_1 = \{(1, x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}_p\},$$

$$V_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} \mid x_1, x_2 \in \mathbb{Z}_p \right\}.$$

Scalar product

$$(1, x_1, x_2) \begin{pmatrix} y_1 \\ y_2 \\ -1 \end{pmatrix} = y_1 + x_1 y_2 - x_2.$$

The second model of $\mathcal{M}$

Let

$$G' = \left\{ g_{abc} = \begin{pmatrix} 1 & a & b+ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\}.$$

Matrix g_{abc} is invertible, and

$$g_{abc}^{-1} = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

The second model of $\mathcal{M}$

Proposition 4

The set G' together with the operation of matrix-multiplication form a group, which is isomorphic to $(\mathbb{Z}_p)^2 \rtimes \mathbb{Z}_p$, and, in fact, it is the Sylow subgroup of $SL(3,p)$.

Define an action of G' on $\Omega = V_1 \cup V_2$ by:

$$x^g = \begin{cases} x \cdot g & \text{if } x \in V_1 \\ g^{-1} \cdot x & \text{if } x \in V_2, \end{cases} \quad \text{for all } g \in G'.$$

$$x^g \cdot y^g = (x \cdot g) \cdot (g^{-1} \cdot y) = x \cdot gg^{-1} \cdot y = x \cdot y.$$

The second model of $\mathcal{M}$

Proposition 5

Groups G and G' are isomorphic.

Corollary 6

The scalar product defined in the second model is invariant with respect to G'. Thus, all association schemes may be redefined in these new terms.

Some observations and recent proofs

Observation 7

Association schemes $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ are algebraic mergings of $\mathcal{M}$.

Observation 8

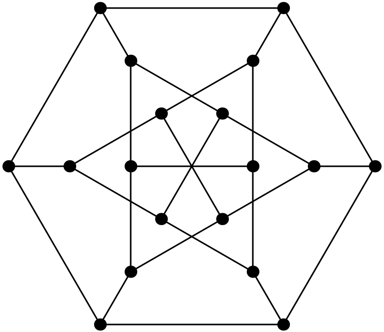
$$\mathrm{AAut}(\mathcal{M}) \cong (\mathbb{Z}_{p-1}^2 \rtimes \mathbb{Z}_2) \times AGL(1,p).$$

Theorem 9

$$\mathrm{AAut}(\mathcal{M}_1) \cong \mathbb{Z}_{p-1}^2.$$

Links to other combinatorial structures

The graph defined by the relation U_0 for $p = 3$ is the Pappus graph.



Links to other combinatorial structures

McKay-Miller-Širáň graphs

Let p be an odd prime and put $V_p = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_p$ as vertex set of H_p. Let ω be a primitive element.

- If $p = 4r + 1$ then define $X = \{1, \omega^2, \omega^4, \ldots, \omega^{p-3}\}$, $X' = \{\omega, \omega^3, \ldots, \omega^{p-2}\}$.
- If $p = 4r + 3$ then define $X = \{\pm 1, \pm\omega^2, \ldots, \pm\omega^{2r}\}$, $X' = \{\pm\omega, \pm\omega^3, \ldots, \pm\omega^{2r+1}\}$.

Links to other combinatorial structures

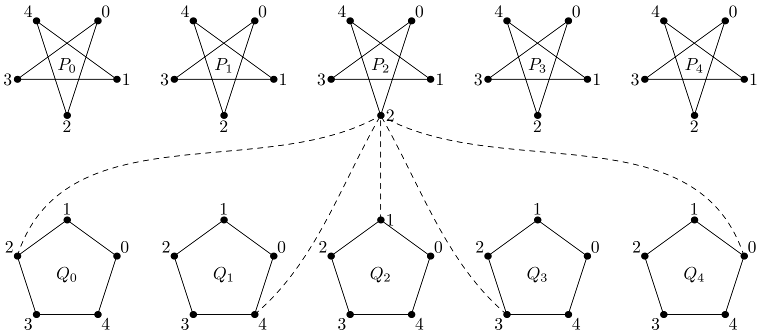
The adjacency in the graph H_p is defined as follows:

$$\begin{array}{rcl} (0,x,y) \text{ is adjacent to } (0,x,y') & \text{if and only if} & y-y' \in X, \\ (1,k,q) \text{ is adjacent to } (1,k,q') & \text{if and only if} & q-q' \in X', \\ (0,x,y) \text{ is adjacent to } (1,k,q) & \text{if and only if} & y = kx+q. \end{array}$$

$$H_p = E_0 \cup F_0 \cup \bigcup_{i\in X} A_i \cup \bigcup_{j\in X'} C_j.$$

H_5 is the well-known Hoffman-Singleton graph.

Links to other combinatorial structures



Adjacencies are between i in P_j and $i \oplus jk$ in Q_k for all $0 \leq i, j, k \leq 4$. (Robertson)

Links to other combinatorial structures

Wenger graphs

The graph $W_n(q)$ has as vertex set two copies $\mathcal{P}$ and $\mathcal{L}$ of the $(n+1)$-dimensional vector space over $\mathbb{F}_q$. The adjacency between points $P = [p_1, \ldots, p_{n+1}]$ and "lines" $L = (l_1, \ldots, l_{n+1})$ is given by the system:

$$\begin{aligned} l_2 + p_2 &= p_1 l_1 \\ l_3 + p_3 &= p_1 l_2 \\ &\vdots \\ l_{n+1} + p_{n+1} &= p_1 l_n. \end{aligned}$$

Links to other combinatorial structures

$n = 1$

Wenger graphs $W_1(p)$ are isomorphic to the graphs defined by U_0.

For example, $W_1(3)$ is isomorphic to the Pappus graph.

Wenger graphs belong to a richer family of graphs defined by a system of equations.

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Thank you

Thank you for your attention.