Symmetries of Latin Squares

Ian Wanless

Monash University, Australia

Joint work with Brendan McKay, Doug Stones,
Petr Vojtěchovský and Xiande Zhang

Latin squares

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e.g. $\begin{pmatrix} 3 & 6 & 1 & 2 & 5 & 4 \\ 4 & 1 & 2 & 3 & 6 & 5 \\ 5 & 2 & 3 & 4 & 1 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \\ 1 & 4 & 5 & 6 & 3 & 2 \\ 2 & 5 & 6 & 1 & 4 & 3 \end{pmatrix}$ is a Latin square of order 6.

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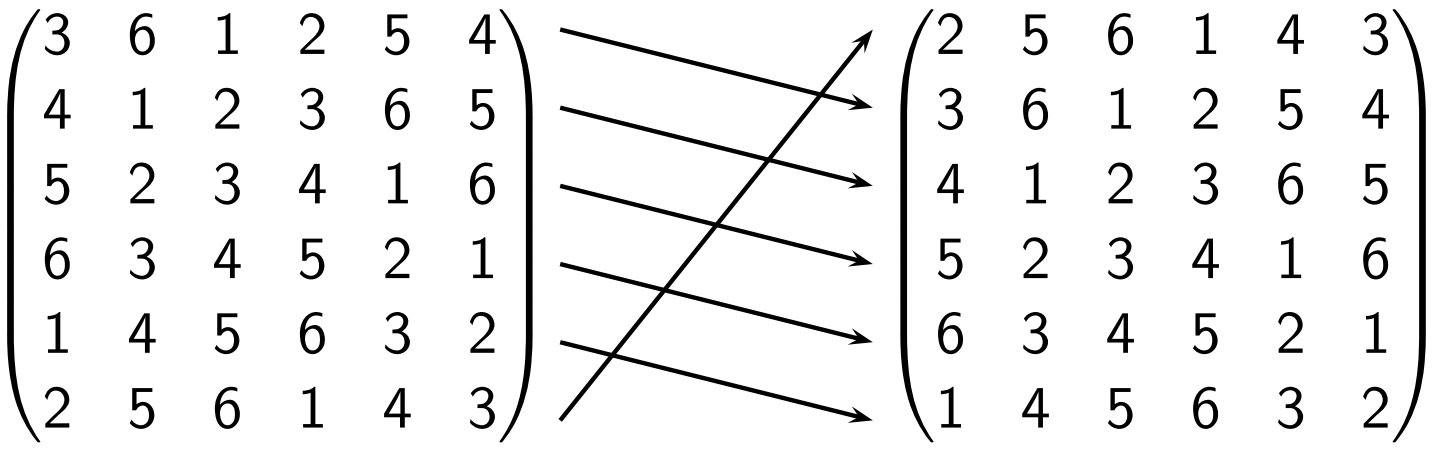
Transformations and Symmetries

A *symmetry* of a Latin square is a transformation that leaves it unchanged.



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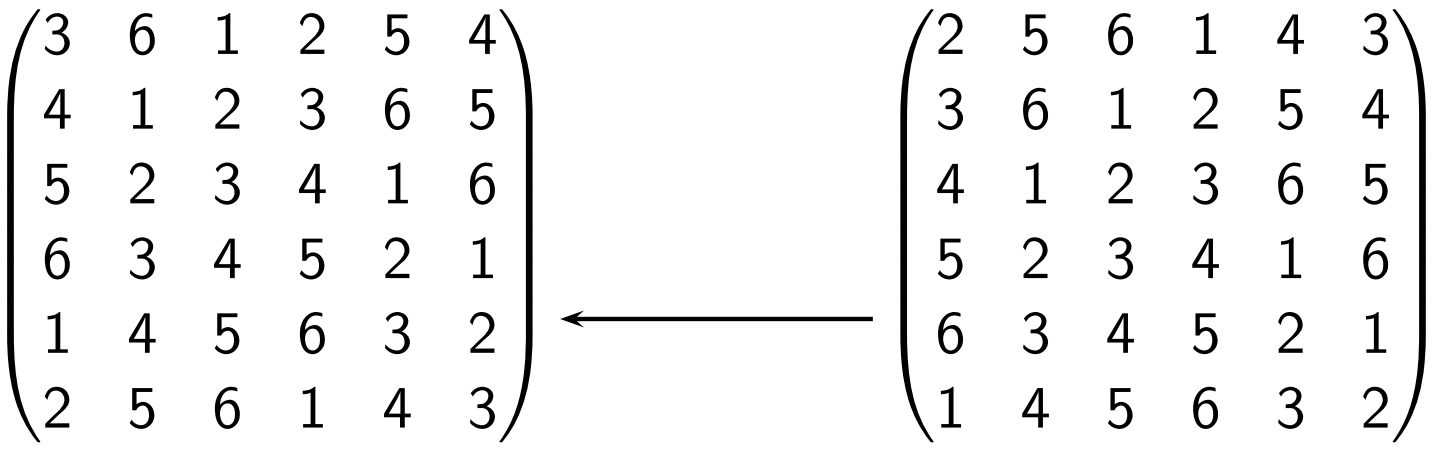

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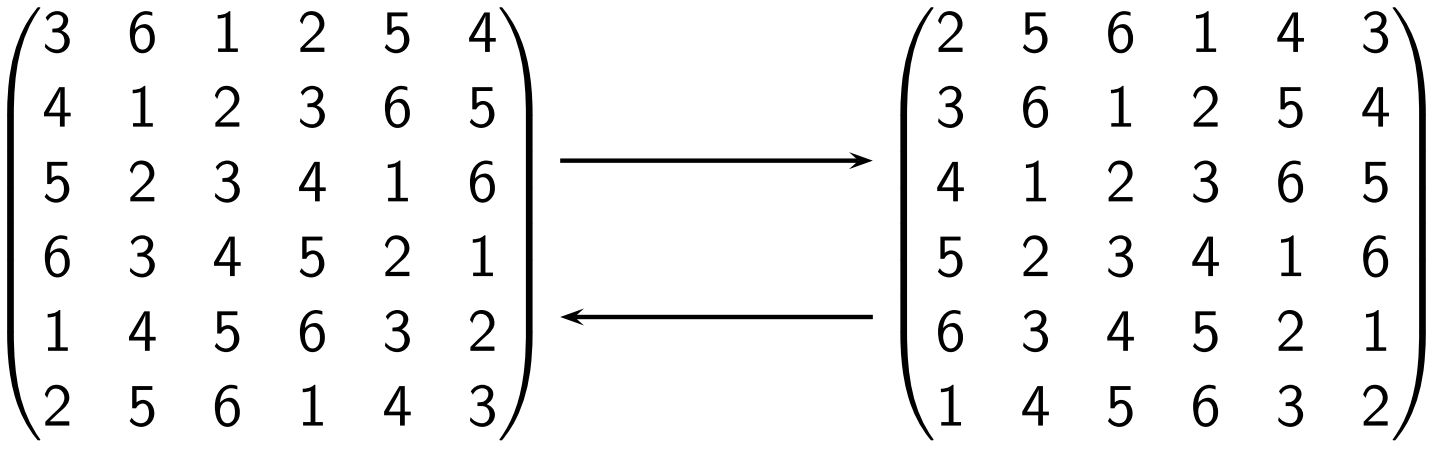

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The combination of these two transformations is a symmetry of our Latin square.

Another example

$$\begin{pmatrix} 4 & 1 & 3 & 5 & 2 \\ 3 & 5 & 2 & 4 & 1 \\ 2 & 4 & 1 & 3 & 5 \\ 1 & 3 & 5 & 2 & 4 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$$

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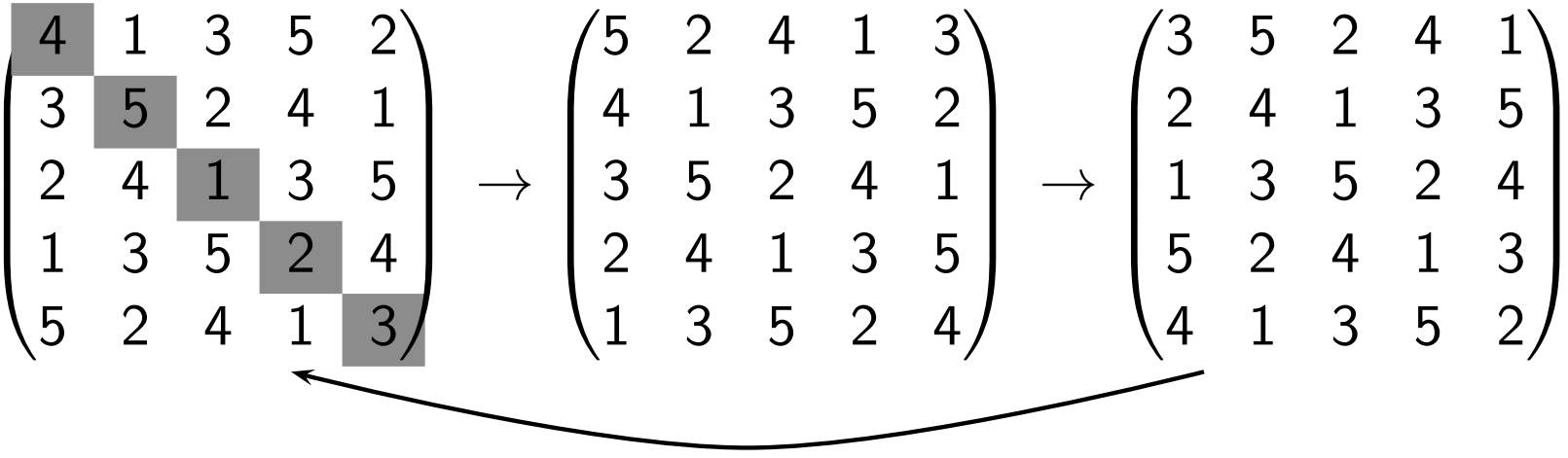

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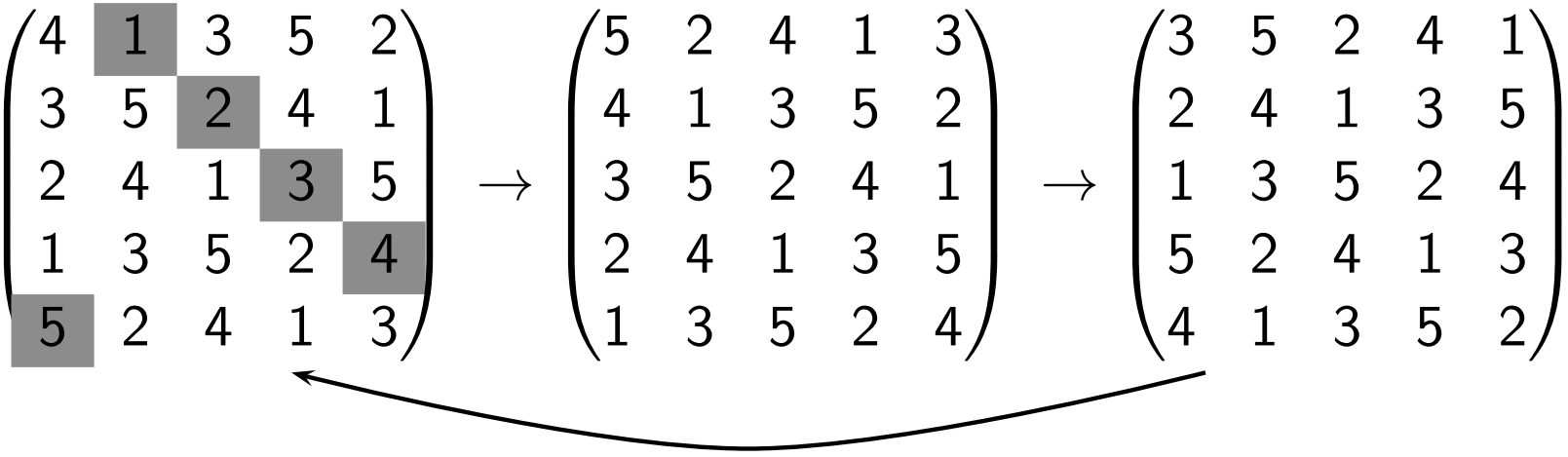

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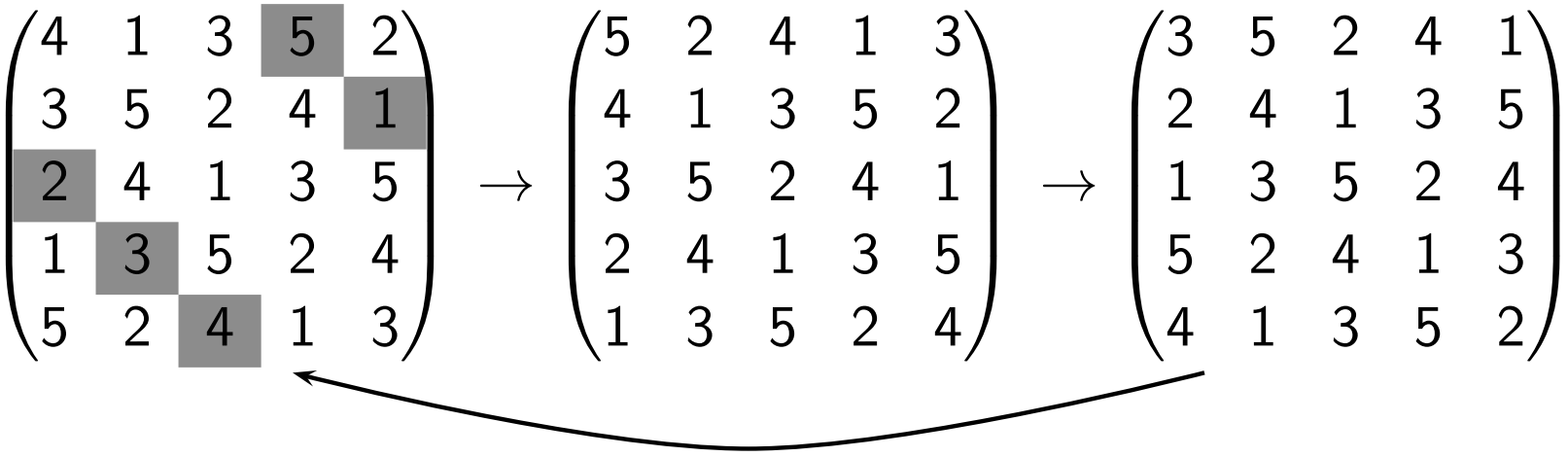

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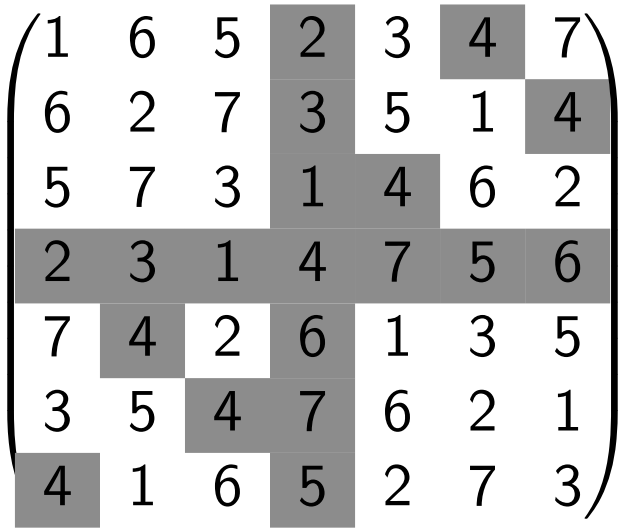

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More generally, a symmetry which permutes rows, columns and symbols (but might apply different permutations to each), is called an *autotopism*.

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These earlier examples showed that $(12345) \in \mathrm{aut}(5)$ and $(123)(4)(567) \in \mathrm{aut}(7)$.

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This talk reports our attempts to understand $\mathrm{aut}(n)$ and $\mathrm{atp}(n)$.

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- is connected to the character theory of quasigroups

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And the fact that $\big((123456), \varepsilon, (123456)\big) \in \mathrm{atp}(6)$, implies that $\big((142536), (231546), \varepsilon\big) \in \mathrm{atp}(6)$.

Number of possible cycle structures

n	3 diff	2 diff	$\#\mathrm{aut}(n)$	$\#\mathrm{atp}(n)$
1			1	1
2		1	1	2
3		1	3	4
4		5	4	9
5		1	5	6
6	1	11	6	18
7		1	9	10
8		25	12	37
9		10	13	23
10	1	23	14	38
11		1	18	19
12	7	113	26	146
13		1	24	25
14	1	37	24	62
15	1	34	39	74
16		151	50	201
17		1	38	39

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9		10	13	23	
10	1	23	14	38	
11		1	18	19	⟵
12	7	113	26	146	
13		1	24	25	⟵
14	1	37	24	62	
15	1	34	39	74	
16		151	50	201	
17		1	38	39	⟵

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We also had theorems giving general constructions for many cases. In particular, we categorised all automorphisms with three or fewer non-trivial cycles.

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Then $\alpha \in \mathrm{aut}(n)$ iff one of the following holds:

1. $d_1 = d_2 = d_3$ and (a) $d_\infty \leqslant 3d_1$ and (b) if d_1 is even then $d_\infty \geqslant 1$,
2. $d_1 > d_2 = d_3$ and (a) $d_1 \geqslant 2d_2 + d_\infty$, (b) d_2 divides d_1, (c) $d_\infty \leqslant 2d_2$, and (d) if d_2 is even and d_1/d_2 is odd then $d_\infty > 0$,
3. $d_1 = d_2 > d_3$ and (a) d_3 divides d_1, (b) $d_\infty \leqslant d_3$, and (c) if d_3 is even then $d_\infty > 0$,
4. $d_1 > d_2 > d_3$ and (a) $d_1 = \mathrm{lcm}(d_2, d_3)$, (b) $d_3 \geqslant d_\infty$, and (c) if d_1 is even then $d_\infty > 0$,
5. $d_1 > d_2 > d_3$ and (a) d_3 divides d_2 which divides d_1, (b) $d_3 \geqslant d_\infty$, and (c) if d_3 is even then $d_\infty > 0$.

The order of an autotopism

The *order* of an autotopism (α, β, γ) is the smallest power m for which $(\alpha, \beta, \gamma)^m = (\varepsilon, \varepsilon, \varepsilon)$. (This is just the least common multiple of all of the cycle lengths in α, β and γ).

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By computer we found that there are no Latin squares of order $n \leqslant 1000$ that have an automorphism of order $m > n$.

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The only problem is it's a bit big to include in the paper!

Prime orders

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Theorem: Suppose L is a Latin square of order n and that (α, β, γ) is an autotopism of L of order m. If k is a prime power divisor of m then

(a) $\mathcal{C}(\alpha, k) = \mathcal{C}(\beta, k) = \mathcal{C}(\gamma, k) \geqslant \frac{1}{2}n$; or

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This is a strong restriction. For prime $n \leqslant 29$ it only leaves

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Is it possible for prime order to have 3 different cycle structures?

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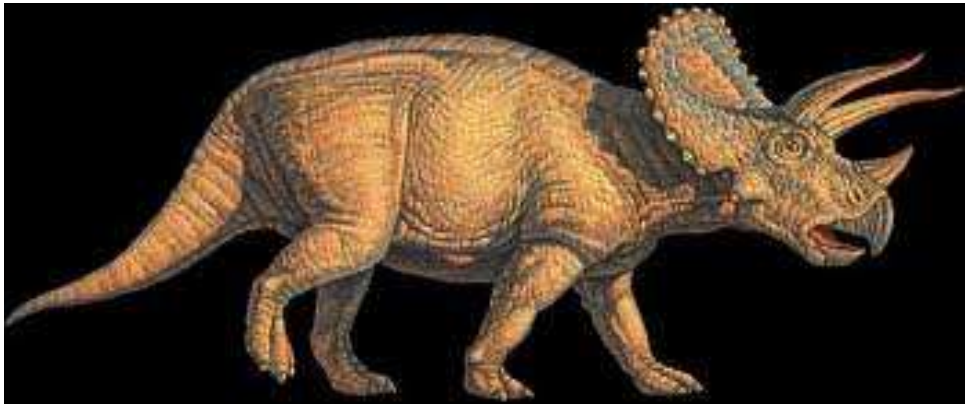

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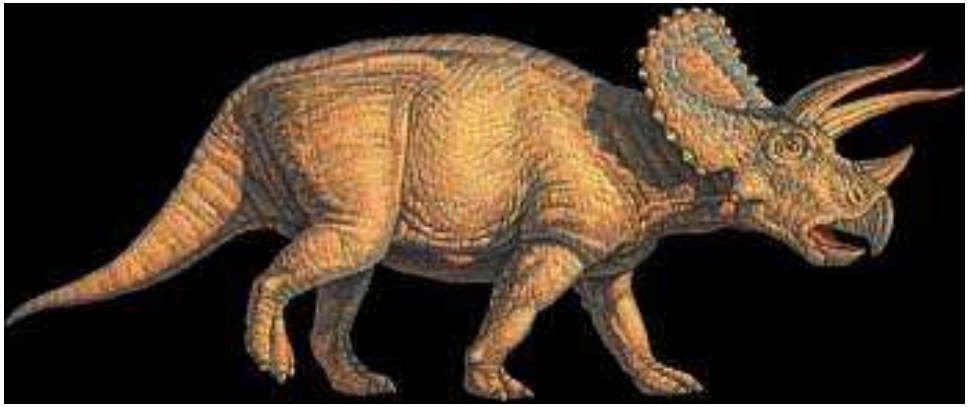

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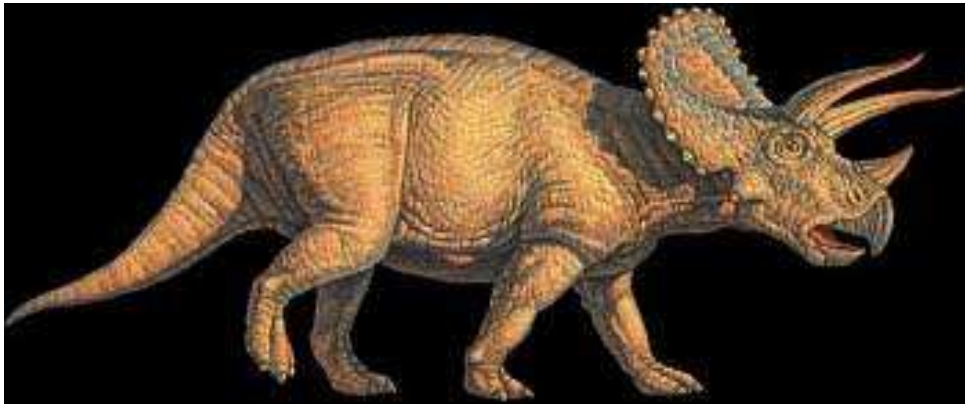

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The cycle structures of the three permutations are
$[30^1, 15^2, 10^3, 6^6, 5^1]$,
$[30^2, 10^3, 6^1, 5^1, 3^{10}]$ and
$[30^2, 15^2, 6^1, 5^1, 2^{15}]$.

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J. Browning, D. S. Stones and I. M. Wanless,
Bounds on the number of autotopisms and subsquares of a Latin square, *Combinatorica*, to appear.

... gives $n^{O(\log n)}$ bounds on the number of different autotopisms a single Latin square of order n can have.

McKay, Meynert and Myrvold 2007

Theorem: Let L be a Latin square of order n and let (α, β, γ) be a nontrivial autotopism of L. Then either

(a) α, β and γ have the same cycle structure with at least 1 and at most $\left\lfloor \frac{1}{2}n \right\rfloor$ fixed points, or

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Corollary: Suppose Q is a quasigroup of order n and that $\alpha \in \mathrm{aut}(Q)$ with $\alpha \neq \varepsilon$.

1. If α has a cycle of length $c > n/2$, then $\mathrm{ord}(\alpha) = c$.
2. If p^a is a prime power divisor of $\mathrm{ord}(\alpha)$ then $\mathcal{C}(\alpha, p^a) \geqslant \frac{1}{2} n$.

(Here $\mathcal{C}(\alpha, k)$ is #points that appear in cycles of α for which the cycle length is divisible by k.)

Some simple cases

Autotopisms where one component is the identity ε:

Theorem: $(\alpha, \beta, \varepsilon) \in \mathrm{atp}(n)$ iff both α and β consist of n/d cycles of length d, for some divisor d of n.

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Corollary: Suppose 2^a is the largest power of 2 dividing n, where $a \geqslant 1$. Suppose each cycle in α, β and γ has length divisible by 2^a. Then $(\alpha, \beta, \gamma) \notin \mathrm{atp}(n)$.

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